AN ALTERNATIVE MATHEMATICAL MODEL FOR SPECIAL RELATIVITY

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ABSTRACT. We present a mathematical model for a physical theory that is compatible with Einstein's Special Relativity Theory. Our model consists of three pseudo-complex dimensions, representing three real dimensions of space, dual to what could be interpreted as three real dimensions of time. We use the term "pseudo-complex", since the mathematical object representing each paired space-time coordinate in our model, is a two-dimensional space over $\mathbb{R}$ whose multiplication makes it into a non-commutative non-associative algebra. We have been unable to find records of this very elementary mathematical object in the literature.

1. Introduction

In Einstein's Special Relativity Theory, a physical event may be represented by a 4-dimensional vector of the form

$$\vec{\chi} = \begin{pmatrix} x \\ y \\ z \\ cti \end{pmatrix}.$$

Passing to another frame of reference is done using the Lorentz transformation, which is given by a 4×4 matrix Λ (see below), by $\vec{\chi}\,' = \Lambda\vec{\chi}$. In our proposed alternative model, a physical event is represented by a vector of the form

$$\vec{x} = \begin{pmatrix} xe + t_x i \\ ye + t_y i \\ ze + t_z i \end{pmatrix}.$$

While x, y, and z are the regular space coordinates, t_x, t_y, and t_z are derived from the single time coordinate t, and satisfy $(t_x^2 + t_y^2 + t_z^2) = t^2$. Each pair of space and time coordinates are then coupled into one "pseudo-complex" coordinate. More precisely, e and i form a basis for a two dimensional vector space over $\mathbb{R}$, which we denote by $\mathbb{M}$, whose elements are thus of the form $\alpha e + \beta i$. We call $\mathbb{M}$ "pseudo-complex"

since we endow it with a multiplication differing from the one in $\mathbb{C}$, by defining

$$ee = e, \ ei = i, \ ie = -i, \ ii = -e.$$

$\mathbb{M}$ turns out to be non-commutative and non-associative. A brief account of the underlying mathematics of $\mathbb{M}$ appears in Section 2.

Section 3 introduces Special Relativity in terms of our proposed model. The Lorentz transformation in our model is given by a matrix L in $\mathbb{M}_{3\times3}$ (see below), constructed so that the transformation $\vec{x}\,' = L \cdot \vec{x}$ is consistent with the regular Lorentz transformation above. We then demonstrate other known physical entities such as the current-charge density vector, the electromagnetic field tensor, and the angular momentum tensor.

In section 4 we present physical results obtained using our model. We first introduce two assumptions, that are motivated by the $\mathbb{M}$ mathematics. The assumptions are that a vector field $\vec{f}$ in $\mathbb{M}^3$, given by

$$\vec{f} = \begin{pmatrix} f_x e + g_x i \\ f_y e + g_y i \\ f_z e + g_z i \end{pmatrix},$$

satisfies

$$\sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial g_i}{\partial t_{x_i}} \right) = 0 \qquad \text{and} \qquad \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial t_{x_i}} + \frac{\partial g_i}{\partial x_i} \right) = 0.$$

Based on these assumptions, we obtain in Theorem 4.1 the continuity equation

$$\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

and the curl Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{and} \qquad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}.$$

In Theorem 4.2 we prove, based on the same assumptions, that the velocity s of a particle in a radial force field F satisfies a certain partial differential equation, namely:

$$\gamma_s \frac{\partial s}{\partial t} + s \frac{\partial \gamma_s}{\partial t} + \frac{\partial \gamma_s}{\partial r} = 0.$$

Finally, in Section 5, we discuss several aspects of our theory which deserve further investigation. These include more $\mathbb{M}$-motivated assumptions at the rank-2 tensor level, that lead to all four Maxwell equations.

2. **The underlying mathematics**

Let $\mathbb{M}$ be a two dimensional vector space over $\mathbb{R}$. Let $\{e, i\}$ be a basis for $\mathbb{M}$. We introduce a multiplication on $\mathbb{M}$ using the bilinear map $\mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{M}$ defined by:

$$ee = e, \ ei = i, \ ie = -i, \ ii = -e.$$

It is clear that $\mathbb{M}$ is non-commutative, since $ei \neq ie$. Simple calculations show that $\mathbb{M}$ is also non-associative and non-unital. Note that the difference between $\mathbb{M}$ and $\mathbb{C}$ can be seen in the definition $ie = -i$ in $\mathbb{M}$, whereas in $\mathbb{C}$, $i1 = i$. Since $\{e, i\}$ is a basis for $\mathbb{M}$, every element $x \in \mathbb{M}$ has a unique representation $x = \alpha e + \beta i$, $\alpha, \beta \in \mathbb{R}$. We will denote $Re(x) := \alpha$, $Im(x) := \beta$, and thus $x = Re(x)e + Im(x)i$. Notice that the bilinearity of the multiplication translates to the distributivity laws:

$$(x + y)z = xz + yz \ \ and \ \ x(y + z) = xy + xz \ \ for \ all \ x, y, z \in \mathbb{M}.$$

We also define a map $* : \mathbb{M} \longrightarrow \mathbb{M}$ by $x^* := xe$. This will play the role of the complex conjugate.

The properties in the following proposition are not difficult to verify:

Proposition 2.1. *For all x, y, z in $\mathbb{M}$:*

1. $\alpha Re(x) = Re(\alpha x), \ \alpha Im(x) = Im(\alpha x), \ \forall \ \alpha \in \mathbb{R}$.
2. e is a left unit element, that is $: ex = x, \ \forall \ x \in \mathbb{M}$.
3. $Re(e)=1, \ Im(e)=0, \ Re(i)=0, \ Im(i)=1, \ Re(0)=0, \ Im(0)=0$.
4. $e^* = e$,
   $i^* = -i$.
5. $x^* = Re(x)e - Im(x)i$.
6. $Re(x^*) = Re(x)$,
   $Im(x^*) = -Im(x)$.
7. $Re(xy) = Re(x)Re(y) - Im(x)Im(y)$,
   $Im(xy) = Re(x)Im(y) - Im(x)Re(y)$.
8. $x(yz) = y(xz)$.
9. $x^*(yz) = (xy)z$.
10. $(xy)z = (zy)x$.
11. $x - x^* = 2Im(x)i$,
    $x + x^* = 2Re(x)e$.
12. $x^{**} = x$.
13. $xy^* = yx^*, \ and \ x^*y = y^*x$.
14. $xy = y^*x^* = (yx)^*$.
15. $Im(x^2) = 0 \ and \ Re(x^2) = Re(x)^2 - Im(x)^2$.

Elements $x \in \mathbb{M}$ with the property that $Re(x) \neq \pm Im(x)$, can be seen to fulfill the condition $Re(x^2) \neq 0$. These play a role of invertible elements, since if we define $x^{-1} := \frac{1}{Re(x^2)}x$, then x^{-1} satisfies the property $x^{-1}x = xx^{-1} = e$ (although e is merely a left unit).

We define matrix multiplication $A \cdot B : \mathbb{M}_{n \times n} \times \mathbb{M}_{n \times n} \longrightarrow \mathbb{M}_{n \times n}$ by

$$(A \cdot B)_{i,j} := (A^*B)_{i,j} = \sum_{k=1}^{n} a_{ik}^* b_{kj}.$$

Likewise, the definition of an $n \times n$ matrix acting on a vector $\vec{x} \in \mathbb{M}^n$ is:

$$(A \cdot \vec{x})_i := (A^* \vec{x})_i = \sum_{k=1}^{n} a_{ik}^* x_k.$$

Under these definitions, matrices are linear operators (over $\mathbb{R}$), and matrix multiplication is consistent with composition of operators, and is therefore associative.

3. Special relativity in terms of the proposed model

We now turn to the Physics. In Einstein's Theory, a physical event is represented by a 4-dimensional vector of the form

$$\vec{\chi} = \begin{pmatrix} x \\ y \\ z \\ cti \end{pmatrix}.$$

Denote the same event, when viewed in a second frame of reference, by

$$\vec{\chi}\,' = \begin{pmatrix} x' \\ y' \\ z' \\ ct'i \end{pmatrix}.$$

Suppose the second frame of reference is moving with velocity $\vec{v} = (v_x, v_y, v_z)$ relative to the first. Then the well known Lorentz transformation Λ relates the two, according to

$$\vec{\chi}\,' = \Lambda\vec{\chi}.$$

Λ is given explicitly by

$$\Lambda = \begin{pmatrix} 1 + (\gamma - 1)\alpha_x^2 & \alpha_x\alpha_y(\gamma - 1) & \alpha_x\alpha_z(\gamma - 1) & \beta\gamma\alpha_x i \\ \alpha_x\alpha_y(\gamma - 1) & 1 + (\gamma - 1)\alpha_y^2 & \alpha_y\alpha_z(\gamma - 1) & \beta\gamma\alpha_y i \\ \alpha_x\alpha_z(\gamma - 1) & \alpha_y\alpha_z(\gamma - 1) & 1 + (\gamma - 1)\alpha_z^2 & \beta\gamma\alpha_z i \\ -\beta\gamma\alpha_x i & -\beta\gamma\alpha_y i & -\beta\gamma\alpha_z i & \gamma \end{pmatrix},$$

where $\alpha_x := \frac{v_x}{v}$, $\alpha_y := \frac{v_y}{v}$, $\alpha_z := \frac{v_z}{v}$, and $v := |\vec{v}| = (v_x^2 + v_y^2 + v_z^2)^{\frac{1}{2}}$. As usual, $\beta := \frac{v}{c}$ and $\gamma := \frac{1}{\sqrt{1-\beta^2}}$.

In our model, a physical event is represented by a three dimensional vector in $\mathbb{M}^3$, of the form

$$\vec{x} = \begin{pmatrix} xe + t_x i \\ ye + t_y i \\ ze + t_z i \end{pmatrix},$$

where $t_x := \alpha_x ct$, $t_y := \alpha_y ct$, and $t_z := \alpha_z ct$. Thus, a physical event in a certain frame of reference A, is not uniquely represented. On the contrary: it's representation depends on a second frame of reference B, moving with velocity $\vec{v}_{A,B} \neq 0$ relative to A. An event in A is represented differently with respect to each frame B.

We replace the usual Lorentz transformation Λ with the following matrix $L \in \mathbb{M}_{3 \times 3}$:

$$L = \begin{pmatrix} [1+(\gamma-1)\alpha_x^2]e & [\alpha_x\alpha_y(\gamma-1)]e & [\alpha_x\alpha_z(\gamma-1)]e \\ -[\beta\gamma\alpha_x^2]i & -[\alpha_x\alpha_y\beta\gamma]i & -[\alpha_x\alpha_z\beta\gamma]i \\ [\alpha_x\alpha_y(\gamma-1)]e & [1+(\gamma-1)\alpha_y^2]e & [\alpha_y\alpha_z(\gamma-1)]e \\ -[\alpha_x\alpha_y\beta\gamma]i & -[\beta\gamma\alpha_y^2]i & -[\alpha_y\alpha_z\beta\gamma]i \\ [\alpha_x\alpha_z(\gamma-1)]e & [\alpha_y\alpha_z(\gamma-1)]e & [1+(\gamma-1)\alpha_z^2]e \\ -[\alpha_x\alpha_z\beta\gamma]i & -[\alpha_y\alpha_z\beta\gamma]i & -[\beta\gamma\alpha_z^2]i \end{pmatrix}.$$

We define

$$\vec{x}\,' := L \cdot \vec{x}.$$

By calculating the components explicitly and comparing, one can verify that $\vec{x}\,'$ is consistent with the usual Lorentz transformation result $\vec{\chi}\,'$, that is:

$$\vec{x}\,' = \begin{pmatrix} x'e + t_x'i \\ y'e + t_y'i \\ z'e + t_z'i \end{pmatrix},$$

where $t_x' = \alpha_x ct'$, $t_y' = \alpha_y ct'$, and $t_z' = \alpha_z ct'$. Note though that $\alpha_x, \alpha_y, \alpha_z$ in $\vec{x}\,'$ fit the original velocity $\vec{v}_{A,B}$. That is, if the regarded physical event would have been represented in the frame B with respect to the frame A (which is moving with velocity $\vec{v}_{B,A} = -\vec{v}_{A,B}$ relative to B), then the representation would be $(\vec{x}\,')^*$ rather than $\vec{x}\,'$.

We further demonstrate our model using examples of other physical entities such as the current-charge density vector, the electromagnetic field tensor, and the angular momentum tensor.

The current-charge density vector is classically represented by

$$\vec{d} = \begin{pmatrix} j_x \\ j_y \\ j_z \\ \rho i \end{pmatrix}.$$

$\vec{d}\,'$ is then obtained by $\vec{d}\,' = \Lambda \vec{d}$.

In our model the current-charge density vector is represented by

$$\vec{\delta} = \begin{pmatrix} j_x e + \alpha_x \rho i \\ j_y e + \alpha_y \rho i \\ j_z e + \alpha_z \rho i \end{pmatrix},$$

and $\vec{\delta}\,' = L \cdot \vec{\delta}$. One can verify that $\vec{\delta}\,'$ is consistent with $\vec{d}\,'$ in the same sense as before.

The electromagnetic field is classically represented by a tensor of rank 2:

$$F = \begin{pmatrix} 0 & B_z & -B_y & -E_x i \\ -B_z & 0 & B_x & -E_y i \\ B_y & -B_x & 0 & -E_z i \\ E_x i & E_y i & E_z i & 0 \end{pmatrix},$$

where B_x, B_y, B_z and E_x, E_y, E_z are the magnetic and electric components, respectively. F' is then obtained by $\Lambda F \Lambda^{-1}$.

In our model the electromagnetic field can be represented by a vector in $\mathbb{M}^3$ (a rank 1 tensor), of the following form:

$$\vec{\mathcal{F}} = \begin{pmatrix} [\alpha_x(B_y + B_z) + \alpha_y B_y + \alpha_z B_z]e + \\ [\alpha_x^2(E_y - E_z) + \alpha_y^2 E_y - \alpha_z^2 E_z - \alpha_x \alpha_y E_z + \alpha_x \alpha_z E_y + \alpha_y \alpha_z (E_z - E_y)]i \\ [-\alpha_x B_x + \alpha_y(B_z - B_x) - \alpha_z B_z]e + \\ [-\alpha_x^2 E_x - \alpha_y^2(E_x + E_z) - \alpha_z^2 E_z - \alpha_x \alpha_y E_z - \alpha_x \alpha_z(E_x + E_z) + \alpha_y \alpha_z E_x]i \\ [-\alpha_x B_x - \alpha_y B_y - \alpha_z(B_x - B_y)]e + \\ [\alpha_x^2 E_x + \alpha_y^2 E_y + \alpha_z^2(E_x + E_y) + \alpha_x \alpha_y(E_x + E_y) + \alpha_x \alpha_z E_y - \alpha_y \alpha_z E_x]i \end{pmatrix}.$$

Since $\vec{\mathcal{F}}$ is a vector, $\vec{\mathcal{F}}'$ is obtained by $\vec{\mathcal{F}}' = L \cdot \vec{\mathcal{F}}$. Once again, $\vec{\mathcal{F}}'$ is consistent with F', in the sense that if we denote

$$F' = \begin{pmatrix} 0 & B_z' & -B_y' & -E_x' i \\ -B_z' & 0 & B_x' & -E_y' i \\ B_y' & -B_x' & 0 & -E_z' i \\ E_x' i & B_y' i & B_z' i & 0 \end{pmatrix},$$

then a very tedious calculation proves that $\vec{\mathcal{F}}'$ is exactly

$$\vec{\mathcal{F}}'=\begin{pmatrix} [\alpha_x(B'_y+B'_z)+\alpha_yB'_y+\alpha_zB'_z]e+ \\ [\alpha_x^2(E'_y-E'_z)+\alpha_y^2E'_y-\alpha_z^2E'_z-\alpha_x\alpha_yE'_z+\alpha_x\alpha_zE'_y+\alpha_y\alpha_z(E'_z-E'_y)]i \\ [-\alpha_xB'_x+\alpha_y(B'_z-B'_x)-\alpha_zB'_z]e+ \\ [-\alpha_x^2E'_x-\alpha_y^2(E'_x+E'_z)-\alpha_z^2E'_z-\alpha_x\alpha_yE'_z-\alpha_x\alpha_z(E'_x+E'_z)+\alpha_y\alpha_zE'_x]i \\ [-\alpha_xB'_x-\alpha_yB'_y-\alpha_z(B'_x-B'_y)]e+ \\ [\alpha_x^2E'_x+\alpha_y^2E'_y+\alpha_z^2(E'_x+E'_y)+\alpha_x\alpha_y(E'_x+E'_y)+\alpha_x\alpha_zE'_y-\alpha_y\alpha_zE'_x]i \end{pmatrix}.$$

The electromagnetic field in our model also has a representation as a tensor of rank 2, namely:

$$\mathcal{F}=\begin{pmatrix} 0 & \begin{matrix}B_ze+\\ [\alpha_xE_y-\alpha_yE_x]i\end{matrix} & \begin{matrix}-B_ye+\\ [\alpha_xE_z-\alpha_zE_x]i\end{matrix} \\ \begin{matrix}-B_ze+\\ [\alpha_yE_x-\alpha_xE_y]i\end{matrix} & 0 & \begin{matrix}B_xe+\\ [\alpha_yE_z-\alpha_zE_y]i\end{matrix} \\ \begin{matrix}B_ye+\\ [\alpha_zE_x-\alpha_xE_z]i\end{matrix} & \begin{matrix}-B_xe+\\ [\alpha_zE_y-\alpha_yE_z]i\end{matrix} & 0 \end{pmatrix}$$

As such, $\mathcal{F}'$ is obtained by $\mathcal{F}' = L \cdot \mathcal{F} \cdot L$ (see Remark 5.1 below). Consistency with F' can be verified here too.

Angular momentum is classically represented by the following tensor of rank 2:

$$J=\begin{pmatrix} 0 & xP_y-yP_x & xP_z-zP_x & (xE-P_xt)i \\ yP_x-xP_y & 0 & yP_z-zP_y & (yE-P_yt)i \\ zP_x-xP_z & zP_y-yP_z & 0 & (zE-P_zt)i \\ -(xE-P_xt)i & -(yE-P_yt)i & (zE-P_zt)i & 0 \end{pmatrix},$$

where P_x, P_y, P_z are the components of the angular momentum, and E is the energy. J' is obtained by $J' = \Lambda J \Lambda^{-1}$.

In our model, angular momentum has a representation as a tensor of rank 2, namely:

$$\mathcal{J}=\begin{pmatrix} 0 & \begin{matrix}(xP_y-yP_x)e+\\ [\alpha_y(xE-P_xt)-\\ \alpha_x(yE-P_yt)]i\end{matrix} & \begin{matrix}(xP_z-zP_x)e+\\ [\alpha_z(xE-P_xt)-\\ \alpha_x(zE-P_zt)]i\end{matrix} \\ \begin{matrix}(yP_x-xP_y)e+\\ [\alpha_x(yE-P_yt)-\\ \alpha_y(xE-P_xt)]i\end{matrix} & 0 & \begin{matrix}(yP_z-zP_y)e+\\ [\alpha_z(yE-P_yt)-\\ \alpha_y(zE-P_zt)]i\end{matrix} \\ \begin{matrix}(zP_x-xP_z)e+\\ [\alpha_x(zE-P_zt)-\\ \alpha_z(xE-P_xt)]i\end{matrix} & \begin{matrix}(zP_y-yP_z)e+\\ [\alpha_y(zE-P_zt)-\\ \alpha_z(yE-P_yt)]i\end{matrix} & 0 \end{pmatrix}.$$

Similar to $\mathcal{F}'$, $\mathcal{J}'$ is obtained by $\mathcal{J}' = L \cdot \mathcal{J} \cdot L$, and consistency with J' can be verified. Angular momentum also has a vector representation.

4. **Obtaining physical results using the proposed model**

A vector field $\vec{f}$ in $\mathbb{M}^3$ is given by

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

where each f_i is a function $f_i : \mathbb{M}^3 \longrightarrow \mathbb{M}$. We can also write $\vec{f}$ as

$$\vec{f} = \begin{pmatrix} f_x e + g_x i \\ f_y e + g_y i \\ f_z e + g_z i \end{pmatrix},$$

where now each of $f_x, g_x, f_y, g_y, f_z, g_z$ are thought of as real functions of the variables $x, t_x, y, t_y, z,$ and t_z.

Let $\vec{f}$ be such a vector field. We require the following two assumptions, by which we restrict ourselves to vector fields in $\mathbb{M}^3$ originating from vector fields in $\mathbb{R}^4$ representing known physical entities:

(1) $\vec{f}$ is a function of $t := \sqrt{t_x^2 + t_y^2 + t_z^2}$.
(2) $\vec{f}$ satisfies $L \cdot \vec{f} = \vec{f'}$, such that $\vec{f'}$ is consistent with the usual Lorentz transformation, as in the examples above.

We now add two more assumptions on the vector field $\vec{f}$:

$$(a1) \qquad \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial g_i}{\partial t_{x_i}} \right) = 0.$$

$$(a2) \qquad \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial t_{x_i}} + \frac{\partial g_i}{\partial x_i} \right) = 0.$$

One can verify that both the above sums are invariants of L, i.e.

$$\sum_{i=1}^{3} \left(\frac{\partial f_i'}{\partial x_i'} + \frac{\partial g_i'}{\partial t_{x_i}'} \right) = \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial g_i}{\partial t_{x_i}} \right),$$

and

$$\sum_{i=1}^{3} \left(\frac{\partial f_i'}{\partial t_{x_i}'} + \frac{\partial g_i'}{\partial x_i'} \right) = \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial t_{x_i}} + \frac{\partial g_i}{\partial x_i} \right).$$

The assumptions $(a1)$ and $(a2)$, that all these sums are equal to zero, are motivated by the $\mathbb{M}$ mathematics: A function $f : \mathbb{M} \longrightarrow \mathbb{M}$ can be written as $f(xe + yi) = u(x, y)e + v(x, y)i$ where $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$. If f satisfies a certain differentiability condition, then u and v both have partial derivatives, and the following versions of the Cauchy-Riemann equations hold: $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Our first results deal with Electrodynamics. Clearly, our current-charge density vector and electromagnetic field vector satisfy assumptions (1) and (2). We claim that in inertial systems, by imposing the assumptions $(a1)$ and $(a2)$, we obtain the continuity equation for current-charge density, and the curl Maxwell equations for the electromagnetic field (in vacuum, without sources). We state and prove this claim formally as a Theorem.

Theorem 4.1. *1. Suppose that the current-charge density vector $\vec{\delta}$ satisfies assumption $(a1)$. Then*

$$(4.1) \qquad \nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}.$$

2. Suppose that the electromagnetic field vector $\vec{\mathcal{F}}$ satisfies assumptions $(a1)$ and $(a2)$. Then

$$(4.2) \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$$(4.3) \qquad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}.$$

Proof: 1. Recall that the current-charge density vector is

$$\vec{\delta} = \begin{pmatrix} j_x e + \alpha_x \rho i \\ j_y e + \alpha_y \rho i \\ j_z e + \alpha_z \rho i \end{pmatrix}.$$

We assume that $\vec{\delta}$ satisfies $(a1)$, so that

$$(4.4) \qquad -\sum_{i=1}^{3} \frac{\partial j_{x_i}}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial (\alpha_i \rho)}{\partial t_{x_i}}.$$

The left hand side of the above equation is precisely $-\nabla \cdot \vec{j}$. We need to calculate the right hand side. Since $\alpha_i \rho$ is a function of $t := (\sum_{i=1}^{3} t_{x_i}^2)^{\frac{1}{2}}$,

$$\frac{\partial (\alpha_i \rho)}{\partial t_{x_i}} = \frac{\partial t}{\partial t_{x_i}} \frac{\partial (\alpha_i \rho)}{\partial t}.$$

Now, since

$$(4.5) \qquad \frac{\partial t}{\partial t_{x_i}} = \frac{\partial ((\sum_{i=1}^{3} t_{x_i}^2)^{\frac{1}{2}})}{\partial t_{x_i}} = \frac{1}{2} \frac{1}{(\sum_{i=1}^{3} t_{x_i}^2)^{\frac{1}{2}}} 2 t_{x_i} = \frac{t_{x_i}}{t},$$

it follows that

$$\frac{\partial (\alpha_i \rho)}{\partial t_{x_i}} = \frac{t_{x_i}}{t} \frac{\partial (\alpha_i \rho)}{\partial t}.$$

Since we are dealing with inertial systems, $\frac{\partial \alpha_i}{\partial t} = 0$. Thus,

$$\frac{\partial(\alpha_i \rho)}{\partial t_{x_i}} = \frac{t_{x_i}}{t} \alpha_i \frac{\partial \rho}{\partial t}.$$

If we restrict to points in $\mathbb{M}^3$ where $\frac{t_{x_i}}{t} = \alpha_i$ (for any t there exist such t_{x_i}'s), we obtain

$$\frac{\partial(\alpha_i \rho)}{\partial t_{x_i}} = \alpha_i^2 \frac{\partial \rho}{\partial t}.$$

By summing over i, we find that the right hand side of (4.4) is

$$\sum_{i=1}^{3} \frac{\partial(\alpha_i \rho)}{\partial t_{x_i}} = \sum_{i=1}^{3} \alpha_i^2 \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t}.$$

Thus we obtained the continuity equation (4.1). Note that the result holds for any x, y, z, t, and is not affected by the restriction on the t_{x_i}'s.

2. We first impose the assumption $(a2)$ on our electromagnetic field vector $\vec{\mathcal{F}}$:

(4.6)
$$\sum_{i=1}^{3} \left[\frac{\partial Re(\mathcal{F}_i)}{\partial t_{x_i}} - \frac{\partial Im(\mathcal{F}_i)}{\partial x_i} \right] = 0.$$

From (4.5) we know that $\frac{\partial t}{\partial t_{x_k}} = \frac{t_{x_k}}{t}$. Adding the fact that $\frac{\partial \alpha_i}{\partial t} = 0$, and restricting to points where $\frac{t_{x_k}}{t} = \alpha_k$, we find that

$$\frac{\partial(\alpha_i B_j)}{\partial t_{x_k}} = \frac{\partial \alpha_i}{\partial t_{x_k}} B_j + \alpha_i \frac{\partial B_j}{\partial t_{x_k}} = \frac{\partial \alpha_i}{\partial t} \frac{\partial t}{\partial t_{x_k}} B_j + \alpha_i \frac{\partial B_j}{\partial t} \frac{\partial t}{\partial t_{x_k}} = \alpha_i \alpha_k \frac{\partial B_j}{\partial t}.$$

Also, since $\frac{\partial \alpha_i}{\partial x_m} = 0$, we have

$$\frac{\partial(\alpha_i \alpha_j E_k)}{\partial x_m} = \alpha_i \alpha_j \frac{\partial E_k}{\partial x_m}.$$

We can thus carefully calculate the expression $\sum_{i=1}^{3} \left[\frac{\partial Re(\mathcal{F}_i)}{\partial t_{x_i}} - \frac{\partial Im(\mathcal{F}_i)}{\partial x_i} \right]$ in (4.6) and obtain:

(4.7)
$$\begin{aligned} 0 = \ & (\alpha_x(\alpha_x - \alpha_y - \alpha_z) - 1) \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \frac{\partial B_x}{\partial t} \right) + \\ & (\alpha_y(\alpha_x - \alpha_y - \alpha_z) + 1) \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t} \right) + \\ & (\alpha_z(\alpha_x - \alpha_y - \alpha_z) + 1) \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \frac{\partial B_z}{\partial t} \right). \end{aligned}$$

Denote $A = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \frac{\partial B_x}{\partial t}$, $B = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t}$ and $C = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \frac{\partial B_z}{\partial t}$. By taking $\alpha_x = 1$ (and thus $\alpha_y = \alpha_z = 0$) in (4.7), we obtain $B + C = 0$. But this must then be true for any α_i's. Similarly, the cases $\alpha_y = 1$ and $\alpha_z = 1$ lead respectively to $-A + C = 0$ and $-A + B = 0$.

From these three equations we can conclude that $A = B = C = 0$, thus proving $\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$ as claimed in (4.3).

We now impose the assumption $(a1)$ on the vector $\vec{\mathcal{F}}$:

$$(4.8) \qquad \sum_{i=1}^{3} \left[\frac{\partial Re(\mathcal{F}_i)}{\partial x_i} - \frac{\partial Im(\mathcal{F}_i)}{\partial t_{x_i}} \right] = 0.$$

Following the same arguments as before, we obtain

$$(4.9) \qquad \begin{aligned} 0 = \ & (\alpha_x + \alpha_z) \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} + \frac{\partial E_y}{\partial t} \right) + \\ & (\alpha_x + \alpha_y) \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{\partial E_z}{\partial t} \right) + \\ & (\alpha_y - \alpha_z) \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial t} \right). \end{aligned}$$

Denote $A = \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} + \frac{\partial E_y}{\partial t}$, $B = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{\partial E_z}{\partial t}$ and $C = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial t}$. Taking $\alpha_x = 1$ we get $A + B = 0$. Taking $\alpha_y = 1$ we get $B + C = 0$. And taking $\alpha_z = 1$ we get $A - C = 0$. These three relations don't suffice to obtain that $A = B = C = 0$. We thus resort to the following "trick". We rename the original axes by $\tilde{x} := y$, $\tilde{y} := z$, $\tilde{z} := x$. Equation (4.9) then becomes

$$(4.10) \qquad \begin{aligned} 0 = \ & (\tilde{\alpha}_x + \tilde{\alpha}_z) \left(\frac{\partial \tilde{B}_z}{\partial \tilde{x}} - \frac{\partial \tilde{B}_x}{\partial \tilde{z}} + \frac{\partial \tilde{E}_y}{\partial t} \right) + \\ & (\tilde{\alpha}_x + \tilde{\alpha}_y) \left(\frac{\partial \tilde{B}_y}{\partial \tilde{x}} - \frac{\partial \tilde{B}_x}{\partial \tilde{y}} - \frac{\partial \tilde{E}_z}{\partial t} \right) + \\ & (\tilde{\alpha}_y - \tilde{\alpha}_z) \left(\frac{\partial \tilde{B}_z}{\partial \tilde{y}} - \frac{\partial \tilde{B}_y}{\partial \tilde{z}} - \frac{\partial \tilde{E}_x}{\partial t} \right). \end{aligned}$$

Now by returning to the original axes we get from (4.10)

$$(4.11) \qquad \begin{aligned} 0 = \ & (\alpha_y + \alpha_x) \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} + \frac{\partial E_z}{\partial t} \right) + \\ & (\alpha_y + \alpha_z) \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial t} \right) + \\ & (\alpha_z - \alpha_x) \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} - \frac{\partial E_y}{\partial t} \right). \end{aligned}$$

Note that the terms in (4.11) are precisely those of (4.9), which we denoted by A, B and C. Taking for example $\alpha_x = 1$ in (4.11), we get $-B + A = 0$. Along with the relations obtained before from (4.9), we can now deduce that $A = B = C = 0$, thus proving $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ as claimed in (4.2). This completes the proof of the theorem. $\qquad \square$

We now turn to mechanics. Suppose that F is any radial force field, whose source is at the origin of the laboratory frame of reference A.

The velocity potential field $\vec{\sigma}$, created by F, is given in our model by

$$\vec{\sigma} = \begin{pmatrix} \gamma_s s_x e + \gamma_s \alpha_x i \\ \gamma_s s_y e + \gamma_s \alpha_y i \\ \gamma_s s_z e + \gamma_s \alpha_z i \end{pmatrix},$$

where s_x, s_y, s_z denote the components of the velocity (in A) acquired by a particle m in the field F, $s := (s_x^2 + s_y^2 + s_z^2)^{\frac{1}{2}}$, and $\gamma_s := \frac{1}{\sqrt{1-s^2}}$ (we take $c \equiv 1$). We assume that F is the only force acting, and that m has initial velocity 0. In this case, s_x, s_y, s_z, s and γ_s are functions only of $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and $t = (t_x^2 + t_y^2 + t_z^2)^{\frac{1}{2}}$. Thus $\vec{\sigma}$ satisfies assumptions (1) and (2). We take the frame of reference B to be that of the particle m. Thus $\frac{s_{x_i}}{s} = \frac{v_{x_i}}{v} = \alpha_i$.

In our next theorem, we impose the assumption $(a2)$ on $\vec{\sigma}$, and obtain a certain partial differential equation necessarily satisfied by s. We consequently remark that under slow velocity approximations, gravitational acceleration, which is proportional to $\frac{1}{r^2}$, satisfies the equation.

Theorem 4.2. *Suppose that the velocity potential field $\vec{\sigma}$ of any radial force field F satisfies $(a2)$. Then the velocity s of a particle in the field F satisfies the equation:*

$$(4.12) \qquad \gamma_s \frac{\partial s}{\partial t} + s \frac{\partial \gamma_s}{\partial t} + \frac{\partial \gamma_s}{\partial r} = 0.$$

   Proof: The assumption $(a2)$ on $\vec{\sigma}$ gives

$$(4.13) \qquad \sum_{i=1}^{3} \left(\frac{\partial(\gamma_s s_{x_i})}{\partial t_{x_i}} + \frac{\partial(\gamma_s \alpha_i)}{\partial x_i} \right) = 0.$$

First we calculate the left summand in (4.13):

$$\frac{\partial(\gamma_s s_{x_i})}{\partial t_{x_i}} = \gamma_s \frac{\partial s_{x_i}}{\partial t_{x_i}} + s_{x_i} \frac{\partial \gamma_s}{\partial t_{x_i}} = \gamma_s \frac{\partial s_{x_i}}{\partial t} \frac{\partial t}{\partial t_{x_i}} + s_{x_i} \frac{\partial \gamma_s}{\partial t} \frac{\partial t}{\partial t_{x_i}}$$

We previously calculated (see 4.5) that $\frac{\partial t}{\partial t_{x_i}} = \frac{t_{x_i}}{t}$. As in the proof of Theorem 4.1, if we restrict to points in $\mathbb{M}^3$ where $\frac{t_{x_i}}{t} = \alpha_i$, we obtain

$$\frac{\partial(\gamma_s s_{x_i})}{\partial t_{x_i}} = \alpha_i (\gamma_s \frac{\partial s_{x_i}}{\partial t} + s_{x_i} \frac{\partial \gamma_s}{\partial t}).$$

Since $\frac{s_{x_i}}{s}$ is constant (and equal to α_i), we have

$$0 = \frac{\partial}{\partial t}(\frac{s_{x_i}}{s}) = \frac{1}{s^2}(\frac{\partial s_{x_i}}{\partial t} s - \frac{\partial s}{\partial t} s_{x_i}),$$

and it follows that

$$\frac{\partial s_{x_i}}{\partial t} = \alpha_i \frac{\partial s}{\partial t}.$$

Therefore,

$$\frac{\partial(\gamma_s s_{x_i})}{\partial t_{x_i}} = \alpha_i(\gamma_s \alpha_i \frac{\partial s}{\partial t} + s_{x_i}\frac{\partial \gamma_s}{\partial t}) = \alpha_i(\gamma_s \alpha_i \frac{\partial s}{\partial t} + s\alpha_i \frac{\partial \gamma_s}{\partial t}) = \alpha_i^2(\gamma_s \frac{\partial s}{\partial t} + s\frac{\partial \gamma_s}{\partial t}).$$

Next we calculate the right summand in (4.13). A calculation similar to (4.5) shows that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$. Thus

$$\frac{\partial(\gamma_s \alpha_i)}{\partial x_i} = \alpha_i \frac{\partial \gamma_s}{\partial x_i} = \alpha_i \frac{\partial \gamma_s}{\partial r}\frac{\partial r}{\partial x_i} = \alpha_i \frac{x_i}{r}\frac{\partial \gamma_s}{\partial r}.$$

The particle m moves along the radial line where $\frac{x_i}{r} = \alpha_i$, so that

$$\frac{\partial(\gamma_s \alpha_i)}{\partial x_i} = \alpha_i^2 \frac{\partial \gamma_s}{\partial r}.$$

Returning to equation (4.13), we obtain:

$$0 = \sum_{i=1}^{3}\left(\frac{\partial(\gamma_s s_{x_i})}{\partial t_{x_i}} + \frac{\partial(\gamma_s \alpha_i)}{\partial x_i}\right) = \sum_{i=1}^{3}\alpha_i^2\left(\gamma_s \frac{\partial s}{\partial t} + s\frac{\partial \gamma_s}{\partial t} + \frac{\partial \gamma_s}{\partial r}\right).$$

But $\sum_{i=1}^{3}\alpha_i^2 = 1$ so we obtain equation (4.12) as claimed. $\qquad\square$

Remark 4.3. In non-relativistic mechanics, gravitational force F is given by $F = ma = \frac{mMG}{r^2}$. Thus the acceleration, $a = \frac{\partial s}{\partial t}$ is proportional to $\frac{1}{r^2}$. We will show that subject to non-relativistic approximations, gravitational acceleration of the form $\frac{1}{r^2}$ satisfies the equation we obtained in Theorem 4.2. So we examine the particular case where $s << c \equiv 1$. Since

$$\frac{\partial \gamma_s}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-s^2}}\right) = -\frac{1}{2}\frac{1}{(1-s^2)^{\frac{3}{2}}}(-2s)\frac{\partial s}{\partial t} = s\gamma_s^3 \frac{\partial s}{\partial t},$$

and similarly $\frac{\partial \gamma_s}{\partial r} = s\gamma_s^3 \frac{\partial s}{\partial r}$, we obtain from equation (4.12):

$$0 = \gamma_s \frac{\partial s}{\partial t} + s\frac{\partial \gamma_s}{\partial t} + \frac{\partial \gamma_s}{\partial r} = \gamma_s \frac{\partial s}{\partial t} + s(s\gamma_s^3 \frac{\partial s}{\partial t}) + s\gamma_s^3 \frac{\partial s}{\partial r}.$$

As $s << 1$, we approximate $\gamma_s \approx 1$ and $s^2 \approx 0$. Subject to these approximations, the above equation gives

(4.14) $$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial r} = 0.$$

We want to show that $\frac{\partial s}{\partial t} = k\frac{1}{r^2}$, $k \in \mathbb{R}$, solves (4.14). Potential energy is given by $E_p = -\frac{mk}{r}$, and kinetic energy is given by $E_k = \frac{ms^2}{2}$. Thus energy conservation $E_{k_1} + E_{p_1} = E_{k_2} + E_{p_2}$ gives $\frac{ms_1^2}{2} + \frac{mk}{r_1} = \frac{ms_2^2}{2} + \frac{mk}{r_2}$. It follows that $\frac{s_1^2 - s_2^2}{r_2 - r_1} = \frac{2k}{r_1 r_2}$. Thus $\frac{\partial s^2}{\partial r}(r_0) := \lim_{r \to r_0}\frac{s^2(r) - s^2(r_0)}{r - r_0} = -\frac{2k}{r_0^2}$.

Therefore $s\frac{\partial s}{\partial r} = \frac{1}{2}\frac{\partial s^2}{\partial r} = -\frac{k}{r^2}$. We conclude that $\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial r} = k\frac{1}{r^2} - \frac{k}{r^2} = 0$, thus solving (4.14).

5. General remarks

Remark 5.1. **The Tensor Calculus**

We deliberately refrained from using the tensor formalism usually adapted by physicists. Thus, for example, we wrote explicit matrix representations of rank 2 tensors, such as F, rather than the more common $F^{\mu\nu}$ notation. Moreover, we didn't develop the tensor calculus in the $\mathbb{M}$ setting. Our $\mathcal{F}$ undergoing the Lorentz transformation by $\mathcal{F}' = L \cdot \mathcal{F} \cdot L$, for example, is justified by the fact that $\mathcal{F}$ is a contravariant rank 2 tensor.

Remark 5.2. **Assumptions on Anti-symmetric Rank 2 Tensors**

In Theorem 4.1 we assumed that the electromagnetic field vector satisfies $(a1)$ and $(a2)$, and obtained the curl Maxwell equations. We don't possess $\mathbb{M}$-motivated assumptions such as $(a1)$ and $(a2)$ at the vector level, yielding the other two Maxwell equations (in vacuum, with no sources), those of the divergence:

$$\nabla \cdot \vec{B} = 0, \qquad and \qquad \nabla \cdot \vec{E} = 0.$$

Nevertheless, if we turn to the rank 2 tensor representation of the electromagnetic field, we do have similar $\mathbb{M}$-motivated assumption, which we denote $(a3)$ and $(a4)$, that yield both the divergence equations and the curl equations. We prefer to describe this as a remark rather than establish it as a theorem, since our assumptions $(a3)$ and $(a4)$ are not fully justified at this stage.

We write a rank 2 anti-symmetric tensor field as:

$$\mathcal{T} = \begin{pmatrix} 0 & T_{12}e + S_{12}i & T_{13}e + S_{13}i \\ -T_{12}e - S_{12}i & 0 & T_{23}e + S_{23}i \\ -T_{13}e - S_{13}i & -T_{23}e - S_{23}i & 0 \end{pmatrix}.$$

Our new assumptions are

$(a3) \qquad \left(\frac{\partial T_{12}}{\partial z} - \frac{\partial S_{12}}{\partial t_z}\right) - \left(\frac{\partial T_{13}}{\partial y} - \frac{\partial S_{13}}{\partial t_y}\right) + \left(\frac{\partial T_{23}}{\partial x} - \frac{\partial S_{23}}{\partial t_x}\right) = 0.$

$(a4) \qquad \left(\frac{\partial T_{12}}{\partial t_z} - \frac{\partial S_{12}}{\partial z}\right) - \left(\frac{\partial T_{13}}{\partial t_y} - \frac{\partial S_{13}}{\partial y}\right) + \left(\frac{\partial T_{23}}{\partial t_x} - \frac{\partial S_{23}}{\partial x}\right) = 0.$

The assumption that the electromagnetic field tensor $\mathcal{F}$ satisfies $(a3)$, translates (after careful calculations and cancellations) exactly into the divergence Maxwell equation for B:

$$\nabla \cdot \vec{B} = 0.$$

The assumption that $\mathcal{F}$ satisfies $(a4)$, translates into

$$\alpha_x\left(\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}-\frac{\partial B_x}{\partial t}\right)+\alpha_y\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}-\frac{\partial B_y}{\partial t}\right)+\alpha_z\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}-\frac{\partial B_z}{\partial t}\right)=0,$$

from which (by choosing appropriate α_i's, similar to the proof of Theorem 4.1) we obtain the curl Maxwell equation for E:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

The electromagnetic field tensor has another representation (the dual representation), obtained by interchanging the roles of E and B:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & \begin{matrix} -E_z e+ \\ [\alpha_x B_y - \alpha_y B_x]i \end{matrix} & \begin{matrix} E_y e+ \\ [\alpha_x B_z - \alpha_z B_x]i \end{matrix} \\ \begin{matrix} E_z e+ \\ [\alpha_y B_x - \alpha_x B_y]i \end{matrix} & 0 & \begin{matrix} -E_x e+ \\ [\alpha_y B_z - \alpha_z B_y]i \end{matrix} \\ \begin{matrix} -E_y e+ \\ [\alpha_z B_x - \alpha_x B_z]i \end{matrix} & \begin{matrix} E_x e+ \\ [\alpha_z B_y - \alpha_y B_z]i \end{matrix} & 0 \end{pmatrix}$$

The assumption that $\mathcal{F}_1$ satisfies $(a3)$ translates into the divergence Maxwell equation for E:

$$\nabla \cdot \vec{E} = 0.$$

And finally, the assumption that $\mathcal{F}_1$ satisfies $(a4)$ translates into

$$\alpha_x\left(\frac{\partial B_y}{\partial z}-\frac{\partial B_z}{\partial y}-\frac{\partial E_x}{\partial t}\right)+\alpha_y\left(\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}-\frac{\partial E_y}{\partial t}\right)+\alpha_z\left(\frac{\partial B_x}{\partial y}-\frac{\partial B_y}{\partial x}-\frac{\partial E_z}{\partial t}\right)=0,$$

from which (by choosing appropriate α_i's) we obtain the curl Maxwell equation for B:

$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}.$$

Remark 5.3. **An Alternative Model Based on $\mathbb{R}^6$**

The basic idea of interpreting time in three real dimensions, could have equally well been mathematically modelled in $\mathbb{R}^6$, instead of $\mathbb{M}^3$. This could be done by replacing

$$\vec{x} = \begin{pmatrix} xe + ct_x i \\ ye + ct_y i \\ ze + ct_z i \end{pmatrix}$$

with

$$\vec{x} = \begin{pmatrix} x \\ ct_x \\ y \\ ct_y \\ z \\ ct_z \end{pmatrix},$$

and likewise other vectors; Matrices in $\mathbb{M}_{3\times 3}$ should then translate to matrices in $\mathbb{R}_{6\times 6}$ by replacing each $\alpha e + \beta i$ entry with a 2×2 block of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

For example, our Lorentz transformation matrix L would be replaced by

$$\begin{pmatrix} 1+(\gamma-1)\alpha_x^2 & -\beta\gamma\alpha_x^2 & \alpha_x\alpha_y(\gamma-1) & -\alpha_x\alpha_y\beta\gamma & \alpha_x\alpha_z(\gamma-1) & -\alpha_x\alpha_z\beta\gamma \\ -\beta\gamma\alpha_x^2 & 1+(\gamma-1)\alpha_x^2 & -\alpha_x\alpha_y\beta\gamma & \alpha_x\alpha_y(\gamma-1) & -\alpha_x\alpha_z\beta\gamma & \alpha_x\alpha_z(\gamma-1) \\ \alpha_x\alpha_y(\gamma-1) & -\alpha_x\alpha_y\beta\gamma & 1+(\gamma-1)\alpha_y^2 & -\beta\gamma\alpha_y^2 & \alpha_y\alpha_z(\gamma-1) & -\alpha_y\alpha_z\beta\gamma \\ -\alpha_x\alpha_y\beta\gamma & \alpha_x\alpha_y(\gamma-1) & -\beta\gamma\alpha_y^2 & 1+(\gamma-1)\alpha_y^2 & -\alpha_y\alpha_z\beta\gamma & \alpha_y\alpha_z(\gamma-1) \\ \alpha_x\alpha_z(\gamma-1) & -\alpha_x\alpha_z\beta\gamma & \alpha_y\alpha_z(\gamma-1) & -\alpha_y\alpha_z\beta\gamma & 1+(\gamma-1)\alpha_z^2 & -\beta\gamma\alpha_z^2 \\ -\alpha_x\alpha_z\beta\gamma & \alpha_x\alpha_z(\gamma-1) & -\alpha_y\alpha_z\beta\gamma & \alpha_y\alpha_z(\gamma-1) & -\beta\gamma\alpha_z^2 & 1+(\gamma-1)\alpha_z^2 \end{pmatrix}.$$

Our motivation for using $\mathbb{M}^3$, is that the new mathematics seems to inspire the physics. For example, assumptions (a1) and (a2) above, which arise from a pure mathematical point of view, lead to the physical results in Theorems 4.1 and 4.2. Nevertheless, the $\mathbb{M}^3$ model has it's disadvantages:

(1) There is no Lorentz group (see Remark 5.4 below).
(2) We can't express rotations.

It is possible that the $\mathbb{R}^6$ model could overcome these problems. We thus don't rule out the possibility of converting to $\mathbb{R}^6$, or reestablishing our theory based on the interplay between the $\mathbb{R}^6$ and $\mathbb{M}^3$ models, and possibly other "pseudo-complex" mathematical models (see Remark 5.5 below).

Remark 5.4. **Absence of the Lorentz Group**

In our $\mathbb{M}^3$ model, Lorentz transformation matrices cannot be composed. The reason for this is as follows. Suppose that a vector $\vec{x}_{AB}$ represents a physical event in a certain frame of reference A, with respect to a second frame of reference B. Let L_{AB} denote the appropriate Lorentz transformation matrix. Now suppose that C is a third frame of reference. In general, it is meaningless to have L_{BC} act on $L_{AB}(\vec{x}_{AB})$, since $L_{AB}(\vec{x}_{AB})$ is not equal to $\vec{x}_{BC}$ (recall that $\vec{x}_{BC}$ contains the α's

of the relative velocity between B and C). In particular, it is not true that $L_{BC} \circ L_{AB} = L_{AC}$. Thus, our model doesn't give rise to an analog of the Lorentz group.

Remark 5.5. **Other Possible Pseudo-Complex Models**

We restrict our attention to a single space dimension, denoted by the x axis, in order to present the following observation. The regular Lorentz transformation is given by

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma x - \beta\gamma ct \\ -\beta\gamma x + \gamma ct \end{pmatrix}.$$

In our model, we coupled the x and t coordinates into a single "pseudo-complex" ($\mathbb{M}$) coordinate, $xe + cti$. We then replaced the regular Lorentz transformation above with:

$$(\gamma e + \beta\gamma i)(xe + cti) = (\gamma x - \beta\gamma ct)e + (-\beta\gamma x + \gamma ct)i.$$

But in doing so, we made a (seemingly) arbitrary choice of signs in the following expression:

$$(\pm\gamma e \pm \beta\gamma i)(xe + cti) = (\gamma x - \beta\gamma ct)e \pm (-\beta\gamma x + \gamma ct)i.$$

Any other combination of $\pm$ signs above is possible, leading to a different definition of multiplication, and thus a totaly different "pseudo-complex" model. We demonstrate some examples in a multiplication chart:

	+++	+−+	−++	−−+	++−	+−−	−+−	−−−
ee	e	e	$-e$	$-e$	e	e	$-e$	$-e$
ei	i	i	$-i$	$-i$	$-i$	$-i$	i	i
ie	$-i$	i	$-i$	i	i	$-i$	i	$-i$
ii	$-e$	e	$-e$	e	$-e$	e	$-e$	e

The $+++$ choice is our $\mathbb{M}$, while the $+-+$ option for example is a commutative associative algebra with a unit! Note though that it is impossible to recover the regular complex numbers $\mathbb{C}$.

We should point out the fact that all these possible models seem to lead back to the same $\mathbb{R}^6$ model, discussed in Remark 5.3. It is worth investigating mathematically these different possible models, and seeing what they give rise to on the physical side.

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